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TWO-DIMENSIONAL BOUNDARY VALUE PROBLEM OF ELECTROELASTICITY FOR A PIEZOELECTRIC MEDIUM WITH CUTS

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A method is proposed for determining the conjugate mechanical and electrical fields in a piezoelectric medium weakened by tunnel cuts which are, generally speaking, curvilinear. The boundary value problem is reduced to the systems of singular integral equations and linear algebraic relations, connecting the functions sought. The approach developed here is allied to that used in [1]. Problems concerning the rectilinear tunnel cracks at the boundary with a conductor were studied in [2, 3].

1. Formulation of the problem. We use the crystallorgraphic xyz coordinate system to consider an unbounded piezoelectric medium (hexagonal 6mm crystal [4], or polarised ceramics [5]) weakened by tunnel cuts L_j $(j = 1, 2, \ldots, r)$ along the y-axis.

For the case of plane deformation of such a medium in the xoz-plane, the system of solution equations in terms of the stress function φ_1 and electric field potential φ_2 has the form (below we assume that $x = x_1$ and $z = x_3$)

$$l_{11}\varphi_1 + l_{12}\varphi_2 = 0, \quad l_{12}\varphi_1 + l_{22}\varphi_2 = 0 \tag{1.1}$$

$$l_{11} = a_{10}\partial_1^4 + a_{12}\partial_1^2\partial_3^2 + a_{14}\partial_3^4, \quad \partial_1 = \frac{\partial}{\partial x_1}$$

$$l_{12} = l_{21} = a_{21}\partial_1^2\partial_3 + a_{23}\partial_3^3, \quad \partial_3 = \frac{\partial}{\partial x_3}$$

$$l_{22} = a_{20}\partial_1^2 + a_{22}\partial_3^2, \quad a_{10} = s_{33} - s_{13}^2 s_{11}^{-1}$$

$$a_{12} = s_{44} + 2s_{13}(1 - s_{12}s_{11}^{-1}), \quad a_{14} = s_{11} - s_{12}^2 s_{11}^{-1}$$

$$a_{21} = s_{13}d_{13}s_{11}^{-1} - d_{33} + d_{15}, \quad a_{23} = d_{13}(s_{12}s_{11}^{-1} - 1)$$

$$a_{20} = \varepsilon_{11}, \quad a_{22} = \varepsilon_{33} - d_{13}^2 s_{11}^{-1}$$

Here $s_{ik} = s_{ik}^E$, d_{ik}, $\varepsilon_{ik} = \varepsilon_{ik}^T$ are, respectively, the elastic compliance, piezoelectric moduli and the dielectric constants appearing in the equation of state of the medium [5]. The functions φ_1 and φ_2 are related to the components of the mechanical stress tensor and electric field intensity E by the formulas

$$\sigma_x = \partial_3^2\varphi_1, \quad \sigma_z = \partial_1^2\varphi_1, \quad \tau_{xz} = -\partial_1\partial_3\varphi_1 \tag{1.2}$$

$$E_x = -\partial_1\varphi_2, \quad E_z = -\partial_3\varphi_2$$

Let the forces $X_n^\pm$ and $Z_n^\pm$ and the potential $\varphi_2^+ = \varphi_2^- = \varphi_2$ be given

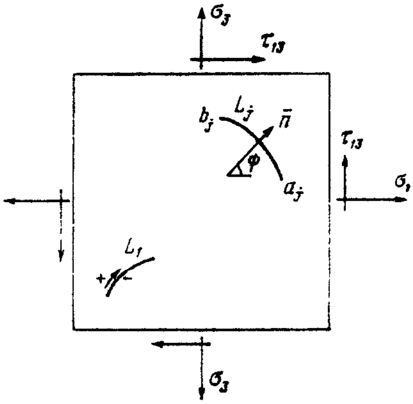

Fig. 1

at the edges L_j , and let a homogeneous field of mechanical stresses σ_1, σ_3 and τ_{13} (see Fig. 1) exist at infinity. We shall assume that L_j are simple, smooth non-intersecting Liapunov curves [6].

Under these conditions, conjugated singular fields of mechanical stresses and of electric field intensity vector will appear in the medium. Our problem consist of describing these fields.

The general solution of the system (1. 1) has the form

$$\varphi_1 = 2\mathrm{Re} \sum_{k=1}^{3} \gamma_k \int \Phi_k(z_k)\,dz_k, \quad \varphi_2 = -2\,\mathrm{Re} \sum_{k=1}^{3} \lambda_k \Phi_k(z_k) \tag{1. 3}$$

$$\gamma_k = a_{20} + a_{22}\mu_k^2, \quad \lambda_k = a_{21}\mu_k + a_{23}\mu_k^3$$

$$z_k = x_1 + \mu_k x_3, \quad \mu_{3+k} = \bar{\mu}_k \quad (k = 1, 2, 3)$$

$$((a_{10} + a_{12}\mu^2 + a_{14}\mu^4)(a_{20} + a_{22}\mu^2) - \mu^2(a_{21} + a_{23}\mu^2)^2 = 0)$$

Here $\Phi_k(z_k)$ are analytic functions of their variables, and the characteristic values μ_k represent the roots of the algebraic equation constained within the brackets. The condition of positive definiteness of the energy functional of the system implies, that $\mathrm{Im}\,\mu_k \neq 0$ $(k = 1, 2, 3)$. We assume in addition that all roots of the equation in question are simple. For example, for the CdSe crystal and the ceramic PZT-5 , for which the numerical values of the constants s_{ik}, d_{ik} and ε_{ik} are given in [5], the computations yield, respectively.

$$\mu_1 = 0.567i, \quad \mu_2 = 0.864i, \quad \mu_3 = 1.825i, \quad \mu_1 = 1.06i$$

$$\mu_2 = -0.258 + 1.084i, \quad \mu_3 = 0.258 + 1.084i$$

Using (1. 2), the equations of state [5] and the representations (1. 3), we find the following expressions for the stresses, displacements u and w , and the electric field intensity in the medium

$$\sigma_x = 2\mathrm{Re} \sum_{k=1}^{3} \gamma_k \mu_k^2 \Phi_k'(z_k), \quad \sigma_z = 2\mathrm{Re} \sum_{k=1}^{3} \gamma_k \Phi_k'(z_k) \tag{1. 4}$$

$$\tau_{xz} = -2\mathrm{Re} \sum_{k=1}^{3} \gamma_k \mu_k \Phi_k'(z_k), \quad u = 2\mathrm{Re} \sum_{k=1}^{3} p_k \Phi_k(z_k)$$

$$w = 2\mathrm{Re} \sum_{k=1}^{3} q_k \Phi_k(z_k), \quad E_x = 2\mathrm{Re} \sum_{k=1}^{3} \lambda_k \Phi_k'(z_k)$$

$$E_z = 2\,\mathrm{Re} \sum_{k=1}^{3} \lambda_k \mu_k \Phi_k'(z_k)$$

$$p_k = a_{14}\gamma_k\mu_k^2 + {}^1/_2(a_{12} - s_{44})\gamma_k - a_{23} - \lambda_k\mu_k$$
$$q_k = {}^1/_2(a_{12} - s_{44})\gamma_k\mu_k + a_{10}\gamma_k\mu_k^{-1} - (a_{21} - d_{15})\lambda_k$$

Taking into account (1.4), we can write the boundary conditions at the contours L_j in the form

$$2\,\mathrm{Re}\left\{\sum_{k=1}^{3} \alpha_{nk}\Phi_k'(t_k)\right\}^{\pm} = W_n^{\pm}(t) \quad (n = 1,2,3) \tag{1.5}$$

$$t_k = \mathrm{Re}\, t + \mu_k \,\mathrm{Im}\, t, \quad t \in L, \quad L = \bigcup_{j=1}^{r} L_j$$

$$\alpha_{1k} = \gamma_k\mu_k a_k(\psi), \quad \alpha_{2k} = \gamma_k a_k(\psi), \quad \alpha_{3k} = \lambda_k a_k(\psi)$$

$$W_1^{\pm} = \mp X_n^{\pm}, \quad W_2^{\pm} = \mp Z_n^{\pm}, \quad W_3^{\pm} = \frac{d\varphi_2}{ds}$$

$$a_k(\psi) = \mu_k \cos\psi - \sin\psi$$

Here ψ is the angle between the normal to the left edge of L_j (when moving from the initial point a_i to the end b_j) and the Ox_1-axis. The first two equations of (1.5) correspond to the mechanical, and the third one $(n = 3)$ to the electrical boundary conditions at the edges of the cut.

To close the system of equations, we must supplement (1.5) with the conditions of uniqueness of the displacements and the electric field potential φ_2.

In this manner we reduce the problem to that of determining the analytic functions $\Phi_k'(z_k)$ in accordance with the boundary conditions (1.5) and some additional conditions which shall be given below.

2. Reduction of the boundary value problem (1.5) to a system of integral and algebraic equations. Let us write the functions $\Phi_k'(z_k)$ in the form

$$\Phi_k'(z_k) = A_k + \frac{1}{2\pi i}\int_L \frac{\omega_k(t)}{t_k - z_k}\,dt_k \quad (k = 1,2,3) \tag{2.1}$$

where the constants A_k satisfy the conditions at infinity

$$2\,\mathrm{Re}\sum_{k=1}^{3}\gamma_k\mu_k^2 A_k = \sigma_1, \quad 2\,\mathrm{Re}\sum_{k=1}^{3}\gamma_k A_k = \sigma_3 \tag{2.2}$$
$$2\,\mathrm{Re}\sum_{k=1}^{3}\gamma_k\mu_k A_k = -\tau_{13}, \quad \mathrm{Re}\sum_{k=1}^{3}\lambda_k A_k = 0$$
$$\mathrm{Re}\sum_{k=1}^{3}\lambda_k\mu_k A_k = 0$$

Passing in (2.1) to the limit and substituting the limiting values into (1.5) we arrive, after manipulations, at the relations

$$2\,\mathrm{Re}\sum_{k=1}^{3}\alpha_{nk}\omega_k(t) = W_n^1(t), \quad W_n^1(t) = W_n^+ - W_n^-, \quad t \in L \tag{2.3}$$

$$2\,\mathrm{Re}\sum_{k=1}^{3}\overset{\circ}{\alpha}_{nk}\left[2A_k+\frac{1}{\pi i}\int_L\frac{\omega_k(t)}{t_k-t_{k0}}dt_k\right]=W_n^{2}(t_0) \qquad (2.4)$$
$$W_n^{2}(t)=W_n^{+}(t)+W_n^{-}(t),\quad t_{k0}=\mathrm{Re}\,t_0+\mu_k\,\mathrm{Im}\,t_0$$
$$t_0\in L\quad(n=1,2,3)\quad(\overset{\circ}{\alpha}_{nk}=\alpha_{nk}(t_0))$$

The relations (2.4) represent a system of three real integral equations in three functions $\omega_k(t)$ which are, in general, complex. The relations (2.4) must be supplemented by three linear algebraic coupling equations (2.3). Taking (2.3) into account, we can write (2.4) in the form

$$\sum_{k=1}^{3}\int_L\{g_{nk}(t,t_0)\,\omega_k(t)+G_{nk}(t,t_0)\,\overline{\omega_k(t)}\}\,dt=\frac{\pi i}{2}N_n(t_0) \qquad (2.5)$$
$$g_{nk}(t,t_0)=\frac{\alpha_{nk}}{t-t_0}+\frac{1}{2}\left[\frac{\overset{\circ}{\alpha}_{nk}}{t_k-t_{k0}}\frac{dt_k}{dt}-\frac{\alpha_{nk}}{t-t_0}\right]\qquad(n,k=1,2,3)$$
$$G_{nk}(t,t_0)=\frac{1}{2}\left[\frac{\overline{\alpha_{nk}}}{t-t_0}-\frac{\overline{\overset{\circ}{\alpha}_{nk}}}{\bar t_k-\bar t_{k0}}\frac{d\bar t_k}{dt}\right]$$
$$N_n(t_0)=\frac{1}{\pi i}\int_L\frac{W_n^{1}dt}{t-t_0}+W_n^{2}+M_n,\quad t_0\in L$$
$$M_1=-2\,(\sigma_1\cos\psi_0+\tau_{13}\sin\psi_0),\quad M_3=0$$
$$M_2=2\,(\tau_{13}\cos\psi_0+\sigma_3\sin\psi_0),\quad \psi_0=\psi(t_0)$$

By virtue of the previous assumption α_{nk} are functions of class H_0 on L [6], the kernels G_{nk} cannot have more than a weak singularity, and the kernels g_{nk} consist of a singular term (Cauchy kernel) and a term with not more than a weak singularity.

The equations (2.3) and (2.5) must be supplemented by the conditions of zero flux of the electric induction vector through any closed contour embracing L_j, and by the condition of uniqueness of the displacements u and w. Taking into account the formulas (1.3), (1.4) and (2.1), we can write these conditions in the form

$$2\,\mathrm{Re}\sum_{k=1}^{3}p_{nk}\int_{L_j}\omega_k(t)\,dt_k=0\quad(n=1,2,3;\ j=1,2,\ldots,r) \qquad (2.6)$$
$$p_{1k}=p_k,\quad p_{2k}=q_k,\quad p_{3k}=\lambda_k a_{20}/\mu_k-\gamma_k d_{15}$$

The equations (2.3), (2.5) and (2.6) determine uniquely the unknown functions $\omega_k(t)$ $(k=1,2,3)$.

3. **Asymptotic formulas for the components of the electric and mechanical fields.** Let us introduce the following parametrization of the contour L_j (below we shall omit the subscript j) by means of the formulas

$$t=t(\beta),\ t_k=t_k(\beta),\ a=t(-1),\ b=t(1),\ (-1\leqslant\beta\leqslant1) \qquad (3.1)$$

We put, in accordance with (3.1) ($\Omega_k^{\circ}(\beta)$ is a function of class H_0 on L)

$$\omega_k(t) = \frac{\omega_k^\circ(t)}{\sqrt{(t-a)(t-b)}} = \frac{\Omega_k^\circ(\beta)}{\sqrt{1-\beta^2}} \tag{3.2}$$

Using (1.4), (2.1), (3.2) and the asymptotic formulas for the Cauchy-type integral near the end of the line of integration [6], we obtain the following asymptotic formulas for the components of the mechanical stress tensor and electric field intensity vector:

$$\sigma_z = S(\gamma, 0), \quad \sigma_x = S(\gamma, 2), \quad \tau_{xz} = -S(\gamma, 1)$$
$$E_x = S(\lambda, 0), \quad E_z = S(\lambda, 1)$$
$$S(\alpha, \beta) = \operatorname{Re} \sum_{k=1}^{3} \frac{1}{\sqrt{2}} \alpha_k \mu_k^\beta \Omega_k^\circ(\pm 1)\, [\mp t_k'(\pm 1)]^{1/2} (z_k - c_k)^{-1/2}$$
$$t_k'(\pm 1) = \left.\frac{t_k}{d\beta}\right|_{\beta=\pm 1}$$

where the upper sign refers to the end of the crack $c = b$, and the lower sign to its beginning $c = a$.

4. Rectilinear crack in a piezoelastic material. Let the material have a single crack occupying the segment $[-l; l]$ of the Ox-axis. In this case the system (2.5) together with the system (2.3), has the form

$$\sum_{k=1}^{3} \frac{2a_{nk}}{\pi i} \int_L \frac{\omega_k(t)}{t-t_0} dt = N_n(t_0) \quad (n = 1, 2, 3) \tag{4.1}$$

Introducing the parametrization $t = \beta l$, $t_0 = \beta_0 l$ and taking into account (3.2), we arrive at three independent equations

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{Q_n^\circ(\beta)}{\sqrt{1-\beta^2}} \frac{d\beta}{\beta - \beta_0} = N_n^*(\beta_0) \quad (n = 1, 2, 3) \tag{4.2}$$
$$Q_n^\circ(\beta) = \sum_{k=1}^{3} 2a_{nk}\Omega_k^\circ(\beta), \quad N_n^*(\beta) = N_n(t)$$

Solving the equations (4.2) we obtain, in the class h_0 [6]

$$Q_n^\circ(\beta_0) = \frac{1}{\pi i} \int_{-1}^{1} \frac{N_n^*(\beta)\sqrt{1-\beta^2}}{\beta - \beta_0} d\beta + i\delta_n \quad (n = 1, 2, 3) \tag{4.3}$$

Separating in (4.3) the real and imaginary parts and using the formulas (2.3), (3.2) and (4.2), we arrive at the conclusion that δ_n are real constants. To compute their values, we must bring in the conditions (2.6). Having found $Q_n^\circ(\beta)$ from (4.3), we obtain the functions $\Omega_k^\circ(\beta)$ from the system of three algebraic equations (4.2), and then, taking into account (3.2), we find the functions (2.1).

Examples. 1°. Let $X_n^\pm = Z_n^\pm = \varphi_2^\pm = 0$ at the crack edges, and let a homogeneous stress field σ_3, τ_{13} exist at infinity. In this case the functions (2.1) have the form (σ denotes the chracteristic stress)

$$\Phi_k'(z_k) = A_k - c_k\sigma\left(i + \frac{z_k}{\sqrt{l^2 - z_k^2}}\right) \quad (k = 1, 2, 3)$$

The values of $\langle c_k\rangle = 10^{-8}\, c_k$, $\langle\sigma_r\rangle = \alpha\sigma_r$ $\langle\sigma_\theta\rangle = \alpha\sigma_\theta$, $\langle\tau_{r\theta}\rangle = \alpha\tau_{r\theta}$, $\langle E_r\rangle = \delta E_r$, $\langle E_\theta\rangle = \delta E_\theta$ $(\alpha = \sqrt{2r}\,/\,(\sigma\sqrt{l}),\ \delta = \varepsilon_{11}\sqrt{2r}\,/\,(\sigma d_{33}\sqrt{l})$ σ_r, σ_θ, $\tau_{r\theta}$, E_r, E_θ representing, respectively, the components of the mechanical stress tensor and the electric field intensity vector, in polar coordinates with the center c situated at the tip of the crack and $r = |z - c|)$ are given for the CdSe crystal in the table. The values in the first line of each block of the table correspond to the case $\sigma = \sigma_3 \neq 0$, $\tau_{13} = 0$ (and we have $\langle c_1\rangle = -155.38i$, $\langle c_2\rangle = 119.53i$, $\langle c_3\rangle = -7.79i$). The values in the second line correspond to the case $\sigma = \tau_{13} \neq 0$, $\sigma_3 = 0$ (and we have $\langle c_1\rangle = -95.45$, $\langle c_2\rangle = 111.46$, $\langle c_3\rangle = -15.31$).

The above data show that a piezoelastic material with a crack acted upon by pure mechanical external forces, develops a strong electric field which leads, in some cases, to a loss of the "electric strength", i.e., to a breakdown of the dielectric.

The conditions for a breakdown are obtained for the case $\sigma_3 \neq 0$, $\tau_{13} = 0$ using [2] and the formulas (4.4) and (1.4). Performing the appropriate computations we obtain

$$\gamma = \pi l\sigma_3^2 \sum_{k=1}^{3} c_k q_k$$

Table 1

θ	0°	30°	60°	90°	120°	150°	180°
$\langle\sigma_r\rangle$	0.93	0.92	1.12	1.14	0.75	0.45	0
	0	0.01	0.03	−0.35	−1.04	−1.77	−1.82
	0	−0.74	−0.42	0.2	−0.01	−0.47	−1.43
$\langle\sigma_\theta\rangle$	0.98	0.88	0.62	0.32	0.12	0.01	0
	0	−0.84	−1.15	−1.11	−0.69	−0.37	−0.25
	0	−0.14	−0.25	−0.25	−0.003	−0.34	0
$\langle\tau_{r\theta}\rangle$	0	0.26	0.39	0.33	0.19	0.06	0
	0.98	0.66	0.25	−0.34	−0.64	−0.47	0
	0	0.12	0.26	0.11	−0.186	−0.17	0
$\langle E_r\rangle$	0	−2.46	−0.41	−0.41	0	−0.41	0
	0	−0.41	−1.23	−1.64	−1.64	−1.23	0
	−1	−0.97	−0.88	−0.74	−0.53	−0.29	0
$\langle E_\theta\rangle$	0	−6.54	−1.23	−0.41	−1.23	0	0
	0	−2.46	−2.46	−1.64	−0.82	−2.05	0
	0	0.22	0.45	0.68	0.883	0.77	1.1

2° Let now $\sigma_1 = \tau_{13} = \sigma_3 = 0$. $X_n^\pm = Z_n^\pm = 0$, $\varphi_2^\pm = \varphi_0 x\,/\,l$. In this case the functions (2.1) have the form

$$\Phi_k'(z_k) = A_k - \frac{\varphi_0 c_k^*}{l}\left(i + \frac{z_k}{\sqrt{l^2 - z_k^2}}\right) \quad (k = 1, 2, 3)$$

The values of the quantities $\langle c_k^*\rangle = 10^{-9}c_k$, $\langle\sigma_r\rangle = \alpha_0\sigma_r$, $\langle\sigma_\theta\rangle = \alpha_0\sigma_\theta$, $\langle\tau_{r\theta}\rangle = \alpha_0\tau_{z\theta}$, $\langle E_r\rangle = \delta_0 E_r$, $\langle E_\theta\rangle = \delta_0 E_\theta$ $(\alpha_0 = 41 d_{33}\sqrt{rl}\,/\,(\varphi_0\varepsilon_{11}\sqrt{2})$, $\delta_0 = \sqrt{rl}\,/\,(\varphi_0\sqrt{2}))$ are given for the ceramic PZT — 5 in the third line of each block of the Table 1, and we have $\langle c_1^*\rangle = 52.95$, $\langle c_2^*\rangle = 0.01 - 1.46i$, $\langle c_3^*\rangle = 0.01 + 1.46i$).

The quantities $\langle\sigma_r\rangle, \ldots, \langle E_\theta\rangle$ appearing in the Table are, respectively, of dimension zero, dimension of the stresses N/m^2, dimension of the stress vector N/k, dimension of $\langle c_k \rangle$ Nm^2/k^2 and dimension of $\langle c_k^* \rangle$ N/k.

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